Hardy Spaces and the Corona Problem

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The topic of this talk is the study of analytic functions on the open unit disc $U := \{z \in \mathbb{C} \mid |z| < 1\}$ with moderate growth on its boundary. To be more precise, a function $f \in H(U)$ is said to be of class H^p (where $0 \le p < \infty$) if

$$\|f\|_p := \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}} < \infty$$

and f is said to be of class H^∞ if $\|f\|_\infty := \sup_{0 \le r < 1} \max_{\theta \in \mathbb{T}} |f(re^{i\theta})| < \infty$. I will give an elementary proof of the fact that $(H^p, \|\cdot\|_p)$ is a Banach space for $1 \le p \le \infty$ er et Banach rum and briefly address the problems that arise when $0 \le p < 1$.

I will then move on to discuss the boundary value behavior of such H^p-functions (the Fatou Lemma) and then cover a beautiful result on the identification of the dual space $(H^p)^*$ of H^p. Such a description is useful in solving extremal problems, as we shall see.

The second part of the talk is about the *Corona problem*, formulated by S. Kakutani in 1941, which states that the space of maximal ideals in H^∞ of the form $M_\zeta := \{f \in H^\infty \mid f(\zeta) = 0\}$ is dense in the maximal ideal space of H^∞ with respect to the Gelfand topology. I will sketch the proof that was offered by Lennart Carleson in 1961 of the following equivalent formulation:

The Corona Theorem: Let $f_1, \ldots, f_n$ be functions in H^∞ such that $|f_1(z)| + \cdots + |f_n(z)| \ge \delta$ for all $|z| < 1$ and some $\delta > 0$. Then there exists functions $g_1, \ldots, g_n$ in H^∞ such that $f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) \equiv 1$.

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